Hybrid Digital Transmission Systems Part II: Information Rate of Hybrid Coaxial Cable Systems

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The information rate of a hybrid coaxial cable transmission system using multilevel pulse amplitude modulation is studied, assuming that the additive repeater noise has a flat spectral density and that statistically independent message symbols are transmitted. Questions considered theoretically are: (i) Reduction in information rate when some repeaters in an "all digital repeater" system are replaced by analog repeaters, (ii) Number of digital repeaters required for converting an analog system to digital service, (iii) Information rate versus number of added analog repeaters in a fixed digital repeater section, (iv) System sensitivity to repeater output power and noise spectral density variations, and (v) Bit rate versus baud rate and achieving the greatest bit rate. Curves and tables answer these questions.

It is economical and theoretically optimum to use identical analog repeaters and uniform repeater spacing for the coaxial cable systems considered. The optimum gain-frequency characteristic for the analog repeaters is the same for both analog and digital transmission. Analog cable systems can be adapted directly to hybrid digital service with no compromise in theoretical performance.

I. INTRODUCTION

In Part I,[1] the general problem of optimizing the parameters in a hybrid (combination digital and analog) transmission system was considered. Closed form expressions were obtained for the transmitting, receiving, and analog repeater filters which would minimize the total mean square error at each digital regenerator. In this part these formulas are applied to the important special case where the transmission medium is coaxial cable, under the assumption that the additive repeater noise has a flat spectral density and that statistically in-

dependent message symbols are transmitted. No attempt was made to include practical details of circuit and filter design.

II. A COAXIAL CABLE HYBRID DIGITAL TRANSMISSION SYSTEM

A hybrid digital transmission system is illustrated in Fig. 1. Information symbols $\{a_k\}$ are transmitted from one digital repeater to the next through L analog repeaters. Multilevel pulse amplitude modulation is considered. Each symbol a_k can assume any one of ν equally spaced levels with probability $1/\nu$. The spacing between two adjacent levels is denoted by d. The levels are assumed symmetrically spaced about zero; hence,

$$E[a_k] = 0$$

$$E[a_k^2] = \frac{d^2(\nu^2 - 1)}{12}. \tag{1}$$

Notice that ν can be an odd as well as an even integer. As usual, the a_k's are assumed to be statistically independent.

The a_k's are transmitted sequentially at T second intervals. The baud rate of the system is $1/T$, and the bit rate is

$$R = (1/T) \log_2 \nu \text{ bps}. \tag{2}$$

It is assumed that the input amplifiers of the analog and digital

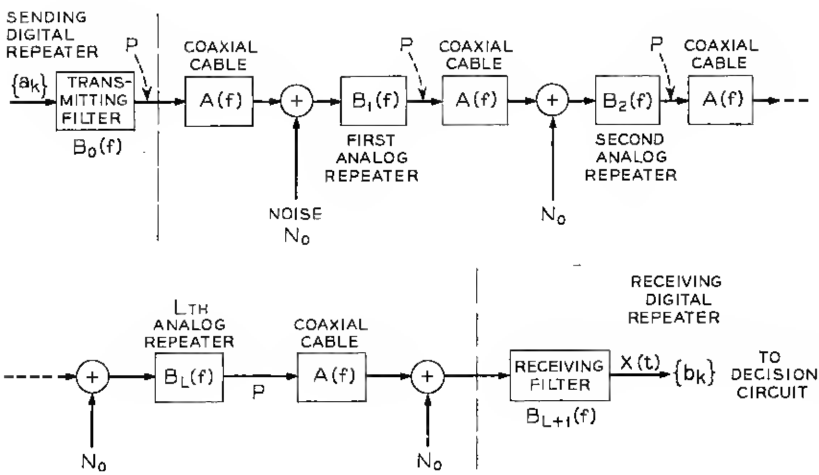

Fig. 1 — A hybrid coaxial cable digital transmission system.

repeaters introduce zero mean thermal noise of constant spectral density N_0 watts per hertz over the frequency band of interest. The average signal powers at the analog and digital repeater outputs are constrained to be P.

As in all linear pulse amplitude modulation systems, the signal at the decision circuit input—$X(t)$ in Fig. 1—is sampled sequentially at T second intervals, and the kth time sample b_k is used as an estimate of a_k. The system's performance is measured by the familiar criterion of mean square error between b_k and a_k, that is, by the quantity

$$\mathcal{E} = E[(b_k - a_k)^2]. \tag{3}$$

The system is said to be optimum if $\mathcal{E}$ is minimized by jointly designing the analog and digital repeater characteristics.

It is shown in Appendix A that for the coaxial cable systems considered the meau square error $\mathcal{E}$ is further minimized if the analog repeaters are equally spaced along the cable. Therefore, uniform repeater spacing is henceforth considered. The transfer function of the coaxial cable between each two repeaters is denoted by $A(f)$ (see Fig. 1). Over most of the useful frequency range one may assume[2]

$$| A(f) | = e^{-S(|f|/f_0)^{\frac{1}{2}}} \tag{4}$$

where

S = cable length in miles

f_0 = frequency at which one mile of cahle has attenuation of one neper (a cable constant).

The analog and digital repeater characteristics that minimize $\mathcal{E}$ can be determined using the general results in Part I (see Appendix B). The main purpose of this part is to explore the interesting characteristics and potentialities of hybrid cable systems. Let us define

$N = (L + 1)N_0$

$\hat{\mathcal{E}}$ = The minimum value of $\mathcal{E}$ that can be attained by jointly designing the analog and digital repeater characteristics.

$$\mu = \frac{S}{(2Tf_0)^{\frac{1}{2}}}. \tag{5}$$

Notice that μ is the cable attenuation in nepers measured at a frequency equal to one half the symbol rate. Using results in Part I, it

is shown in Appendix C that if

$$\frac{P}{N_0} \geqq (L+1)\frac{1}{2T}\frac{1}{\mu^2}[2\mu e^{2\mu} - 3e^{2\mu} + 4e^{\mu} - 1] \tag{6}$$

then the bit rate of the optimum system can be related to all other system parameters by

$$R = \frac{1}{2T}\log_2\left\{1 + \frac{\dfrac{S^2P}{f_0N} + (2\mu - 1)e^{2\mu} + 1}{\dfrac{2}{3\mu^2}[(\mu - 1)e^{\mu} + 1]^2}\cdot\frac{\hat{\mathcal{E}}}{d^2}\right\}. \tag{7}$$

The condition in equation (6) requires that the signal-to-noise ratio of the system be larger tban a certain value. This condition is satisfied under normal operating conditions for the following reason. It can be shown that the quantity

$$\frac{1}{\mu^2}[2\mu e^{2\mu} - 3e^{2\mu} + 4e^{\mu} - 1]$$

in equation (6) is zero when μ is zero, and increases witb μ. Therefore, the right side of equation (6) increases with the number of analog repeaters, the symbol rate $1/T$, the repeater spacing S, and the attenuation in tbe cable (that is, $1/f_0$). It can be shown that if L, $1/T$, S, and $1/f_0$ are made so large that equation (6) is not satisfied, the optimum system will be forced to use a bandwidth less than the Nyquist bandwidth $1/2T$ to reduce thermal noise. A system should not be designed to operate under such an extreme condition since intersymbol interference increases rapidly as the bandwidth is reduced to less than the Nyquist bandwidth $1/2T$.

For equation (7) to be useful one must assume something about the probability distribution of the total interference (intersymbol plus noise). This allows one to relate the ratio $\hat{\mathcal{E}}/d^2$ to the average probability of error. In tbe remainder of this paper tbe natural and useful assumption will be made that tbe total interference is normally distributed. Evidence to date indicates that this assumption is actually conservative and that average error probabilities even less tban those stated would actually be obtained in most cases.[3]

III. SELECTION OF SYMBOL RATE

To facilitate comparing bit rates of systems which have different values, we ignore tbe variation of error probability with number of

levels, which can be at most a factor of two (for details see Ref. 4, pp. 114–118), and assume that it depends only on $\hat{\mathcal{E}}/d^2$. Specifically, a value of $d^2/\hat{\mathcal{E}} = 126$ gives an error rate of 10^{-8} for binary transmission. Unless stated otherwise, this value for the ratio will be used throughout.

Consider now the system parameters. The cable constant f_0 is usually a given parameter. Values which tbe thermal noise spectral density N_0 and the power constraint P may assume are restricted in most cases. As already discussed, if tbe error rate is specified, the ratio $d^2/\hat{\mathcal{E}}$ is also approximately fixed. Thus, the factors over which tbe system designer may exercise the most control are the symbol rate $1/T$, the number of levels ν, the number of analog repeaters L, and the repeater spacing S.

Let us consider the selection of $1/T$.

$$R = (1/T) \log_2 \nu \text{ bps} \tag{2}$$

where $\log_2 \nu$ is the number of bits per symbol. It is proven in Appendix D that under the normal operating condition represented by equation (6), $\log_2 \nu$ decreases when $1/T$ increases. Thus, usually there exists a symbol rate which maximizes the bit rate R. To illustrate this and the significance of selecting the symbol rate, we consider a typical system in the following.

Consider a system using standard ⅜-inch coaxial cable which has

$$f_0 = 5 \times 10^6 \text{ hertz.} \tag{8}$$

The analog or digital repeater output power is constrained to be

$$P = 0.1 \text{ watt.} \tag{9}$$

The thermal noise spectral density depends on the noise figure of the amplifiers. A reasonable assumption* is that

$$N_0 = 1 \times 10^{-19} \text{ watts/hertz} \tag{10}$$

corresponding to a noise figure of 13.8 dB.

Let us assume a repeater spacing S of 1.25 miles. (11)

Consider the case $L = 9$, that is, nine analog repeaters are used between each two digital repeaters. The ratio $d^2/\hat{\mathcal{E}}$ is fixed to 126, corresponding to an error rate of approximately 10^{-8} per 12.5 miles.

If we vary the baud rate $1/T$, we obtain the results in Fig. 2. When the baud rate $1/T$ increases from zero, the bit rate R first increases and then decreases. There is a peak of R at $1/T \cong 2.8 \times 10^8$. Also

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* A thermal noise spectral density of 1.67×10^{-19} watts per hertz was used in Ref. 2 based on a noise figure of about 16.2 dB.

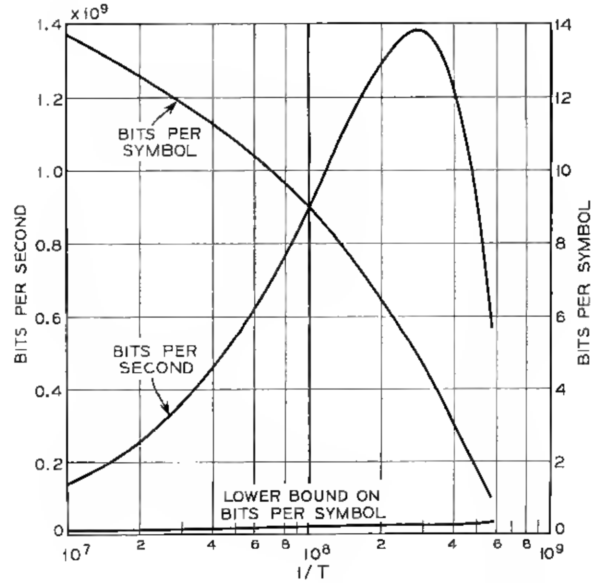

Fig. 2 — Bit rate and number of levels vs symbol rate $1/T$.

shown in Fig. 2 is the quantity "bits per symbol" (that is, $\log_2 \nu$) which, as proven in theory, decreases when the baud rate increases. Notice that the results are meaningful only when ν is an integer. Therefore, one should consider only those points where ν is an integer, or commonly used integers such as 2, 3, 4, 8, and 16.

Two observations are made from Fig. 2:

(*i*) If a low symbol rate such as $1/T \cong 10^7$ is selected, not only is the resulting bit rate too low (about 1/10 of the maximum R), the number of levels must also be extremely large (approximately 2^{14} levels) in order to attain this very low bit rate.

(*ii*) At the maximum bit rate ν is approximately 32, an impractically large number. However, reducing ν to 16, 8, or 4 levels only reduces R from 1.38×10^9 to 1.35×10^9, 1.21×10^9, or 0.96×10^9, respectively.

These observations clearly show the significance of selecting the baud rate and how a baud rate can be chosen for best use of a given system.

Notice that the above results are computed from equations (7) and (2) and that equation (7) is valid if equation (6) holds. By rearranging the terms of equation (6) together with some algebraic

manipulation, it can be shown that equation (7) is valid if the R computed from equation (7) satisfies the inequality

$$R \geqq \frac{1}{2T} \log_2 \left[1 + \frac{6\mu^2}{\mu - 1 + e^{-\mu}} \frac{\hat{\mathcal{E}}}{d^2} \right] \tag{6a}$$

of if ν computed from equations (7) and (2) satisfies the inequality

$$\log_2 \nu \geqq \tfrac{1}{2} \log_2 \left[1 + \frac{6\mu^2}{\mu - 1 + e^{-\mu}} \frac{\hat{\mathcal{E}}}{d^2} \right]. \tag{6b}$$

Notice that equations (6a) and (6b) represent lower bounds on bit rate and bits per baud, respectively. The lower bound on bits per symbol is plotted in Fig. 2. It is seen that, as discussed in Section II, equation (6) is easily satisfied in practice.

IV. REPLACEMENT OF DIGITAL REPEATERS

What happens if some of the repeaters in an all-digital repeater system are replaced by analog repeaters? Because in multilevel transmission analog repeaters might be less expensive than digital repeaters, this may reduce the cost of the system.

Since analog repeaters introduce thermal noise, replacing digital repeaters with analog repeaters decreases the bit rate of the system (assuming a fixed error rate), but the reduction might not be much.

Let us consider the same system specifications (8), (9), (10), and (11), as in Section III, and let us consider three cases:

(*i*) The repeaters in the system are all digital.

(*ii*) 90 percent of the digital repeaters are replaced by analog repeaters (10 percent digital).

(*iii*) 99 percent of the digital repeaters are replaced by analog repeaters (1 percent digital).

The ratio $d^2/\hat{\mathcal{E}}$ is set to 144, 126, and 108, for cases *i*, *ii*, and *iii*, respectively. As discussed in Appendix E, this gives an error rate of approximately 10^{-7} over a distance of 125 miles for all three cases.

Under the above conditions, the bit rates of the three cases are computed using equations (7) and (2). The results are compared in Table I and plotted in Fig. 3.

In comparing cases *i* and *ii*, we see that the bit rate decreases only 18 to 28 percent when 90 percent of the digital repeaters are replaced by analog ones. From cases *i* and *iii*, we see that the bit rate de-

TABLE I— COMPARISON OF BIT RATES

Type of Transmission	Bits per second ($\times 10^9$)		
	Case *i*	Case *ii*	Case *iii*
Binary	0.71	0.58	0.46
Ternary	1.02	0.82	0.63
4-level	1.21	0.96	0.74
8-level	1.56	1.21	0.91
16-level	1.77	1.35	0.98
32-level	1.86	1.38	0.98
64-level	1.85	1.33	0.91

creases 35 to 50 percent when 99 percent of the digital repeaters are replaced by analog ones. The reductions in bit rate are moderate compared with the amount of replacement.

It is important to observe that the bit rate of case *i* is best at 32-level transmission, but it is difficult, if not impossible, to realize a 32-level transmission system. Therefore, one is forced to consider a reduced bit rate. If one uses only digital repeaters, there is only one choice, reducing bits per symbol. However, hybrid systems give another degree of freedom: one may consider various combinations of transmission levels and numbers of analog repeaters.

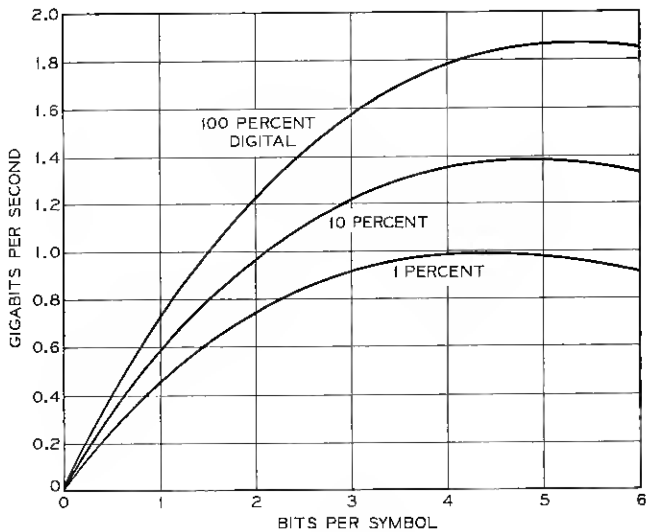

Fig. 3 — Comparison of bit rates of three arrangements.

For example, consider the two choices from Table I:

(*i*) Repeaters all digital, ternary transmission used, bit rate reduced to 1.02×10^9 bits per second.

(*ii*) 90 percent of the repeaters are analog, 4-level transmission, bit rate reduced to 0.96×10^9 bits per second.

Since the bit rates are very close, the selection depends largely on the cost of the repeaters, installation, maintenance, and so on.

V. HYBRID SYSTEM FLEXIBILITIES

We have considered a digital system and computed the reductions in bit rate when digital repeaters are replaced by analog repeaters. But in other applications, the system may be originally built for voice communication with all analog repeaters. As is well known, the analog repeater gain-frequency characteristic for voice communication is shaped to match the loss-frequency characteristic of the coaxial cable.[5] Appendix B shows that such shaping is also optimum for multilevel digital transmission. Thus, an analog system is basically suited for digital transmission provided that digital repeaters are inserted; the question is how many.

Tables such as Table I can be helpful in making such decisions. Case *ii* in Table I corresponds to inserting one digital repeater after every nine analog repeaters, and case *iii* corresponds to inserting a a digital repeater after every ninety-nine analog repeaters. Bit rate can be easily computed for other values of L, ν, S, f_0, P, N_0, and error rate using equations (7) and (2). The results reveal the capacities of various systems.

A hybrid system can be used for either digital or voice communication by installing a digital as well as an analog repeater at the $(L + 1)$st repeater location. The L analog repeaters between can be used for both services without sacrificing the system performance because of common gain-frequency shaping requirements.

VI. INSERTION OF ANALOG REPEATERS

Replacing digital repeaters with analog ones or inserting digital repeaters into an analog system amount to changing the parameter L of a hybrid system. The repeater spacing S is unchanged.

In certain cases, one might wish to fix the distance between two digital repeaters, and vary the number of analog repeaters between. In these cases S varies with L.

Let the distance between two digital repeaters be η miles. Then $S = \eta/L + 1$. Thermal noise spectral density at the input of the receiving digital repeater is $N = (L + 1)N_0$. When L increases, thermal noise increases, but S and cable loss decrease. Hence, the bit rate may increase or decrease depending on thermal noise and cable loss. For instance, if N_0 and f_0 are both very small, increasing L will increase the bit rate; if N_0 and f_0 are large, increasing L will decrease bit rate. The following shows that for typical values of N_0 and f_0, increasing L increases bit rate.

Let $f_0 = 5 \times 10^6$, $P = 0.1$ and $N_0 = 10^{-19}$ as considered earlier, and assume a distance η of 100 miles. Table II shows L for each of the following:

$(1/T)_m$ = The baud rate that maximizes the bit rate
R_{max} = Maximum bit rate in bits per second at $(1/T)_m$
$(\log_2 \nu)_m$ = bits per symbol baud at $(1/T)_m$.

For all L, $d^2/\hat{\varepsilon}$ is set to 126 (error rate $\cong 10^{-8}$ for each digital repeater section). Notice that the table contains values of L which are both impractically small and impractically large, which are included only for completeness.

From Table II, we see that when L increases from 1 to 10, 100, and 1000, R_{max} increases 17, 550, and 12,700 times, respectively. Similarly rapid increases in bit rate are also obtained for η as small as 10 miles or as large as 200 miles. It is concluded that, for the typical values of N_0 and f_0 considered, the insertion of analog repeaters increases the theoretical bit rate rapidly. The number of analog repeaters, how-

TABLE II—BIT RATE VERSUS L FOR η = 100 MILES

L	Bits per second		
	$(1/T)_m$	R_{max}	$(\log_2 \nu)_m$
1	3.68×10^5	2.58×10^6	7.00
2	7.60×10^5	5.10×10^6	6.71
5	2.60×10^6	1.62×10^7	6.22
10	7.53×10^6	4.37×10^7	5.80
20	2.32×10^7	1.24×10^8	5.35
50	1.06×10^8	4.99×10^8	4.73
100	3.31×10^8	1.41×10^9	4.25
200	1.02×10^9	3.85×10^9	3.77
500	4.30×10^9	1.36×10^{10}	3.16
1000	1.22×10^{10}	3.27×10^{10}	2.68

ever, is limited by such practical considerations as misalignment, equalization, and economy.

VII. BIT RATE, POWER, AND NOISE

We have assumed a repeater output power, P, of 0.1 watt and a thermal noise spectral density, N_0, of 10^{-19} watts per hertz. Although these are conservative figures, it is nevertbeless interesting to ask how sensitive tbe system is to variations in P and N_0.

Let us consider the coaxial cable system specified by (8), (11), and $d^2/\mathcal{E} = 126$. Notice from (7) that we need to consider only the ratio P/N_0, not P and N_0 separately.

$$\frac{P}{N_0} = \frac{0.1}{10^{-19}} = 10^{18}.$$

It is very unlikely that P/N_0 will vary by a factor of 10^7, but let us consider such a range.

Sinee $N = (L + 1)\, N_0$ and since L appears only in the ratio P/N in (7), we may vary P/N instead of P/N_0 so that the results ean be used for all L. For instance, when $L = 9$, varying P/N from 10^{18} to 10^{11} corresponds to varying P/N_0 from 10^{19} to 10^{12}.

In Fig. 4, P/N is varied from 10^{18} to 10^{11}. For each P/N, bit rate is shown versus bits per symbol (that is, versus the logarithm of the number of levels of transmission). We see that the reduction in bit rate is moderate compared with variation in P/N. For instance, when P/N reduees from 10^{18} to 10^{16} (by a factor of 10^2), bit rate reduees only 37 percent at binary transmission, or 40 percent at 4-level transmission. Thus, the system can tolerate a reasonable amount of variation in P/N. However, as one should expect, an extremely severe reduction in P/N is not tolerable. For example, if P/N reduces from 10^{18} to 10^{11}, the maximum bit rate would be reduced to 6×10^7.

Figure 4 shows that binary transmission is the least sensitive to variation in P/N. As P/N decreases, the peak of the curve shifts to the left, reducing the theoretical advantage of multilevel transmission over binary.

VIII. CONCLUSIONS

The information rate of a hybrid coaxial cable digital transmission system has been evaluated theoretically. Because of the assumptions made, the various curves involving information rate are to be inter-

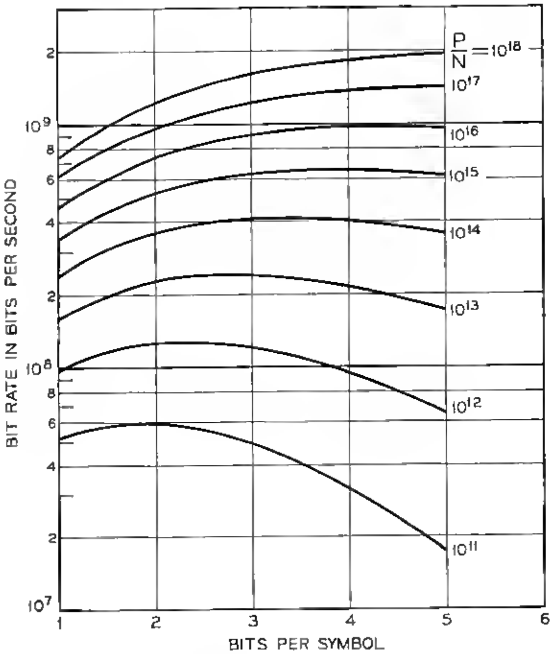

Fig. 4 — Bit rate vs bits per symbol for P/N from 10^{18} to 10^{11}.

preted more in the nature of upper bounds than actual performance curves to be attained in practice. Taken as such, the curves nevertheless illustrate the interesting characteristics and potentialities of hybrid cable systems. Among the more important results of the study are:

(*i*) In general it is not only economical but also optimum to use uniform repeater spacing and identical analog repeaters. Moreover, the optimum gain-frequency characteristic for the analog repeaters is the same for both analog and digital transmission. Therefore, an analog system can be adapted directly to hybrid digital service with no compromise in theoretical performance over the frequency band that the analog repeaters were originally designed for.

(*ii*) In general, hybrid systems give system designers an additional degree of freedom. For example, the curves of Fig. 3 show that, for the particular system illustrated, the sacrifice in theoretical information capacity for binary transmission between a system using all digital repeaters and one in which only one in ten repeaters is digital is about 20 percent. In order to remove low frequency energy from

the transmitted pulse spectrum, it is customary in present binary PCM systems to actually use some form of 3-level transmission, thus incurring a sacrifice of 37 percent of the information capacity. Therefore, another method of solving this dc problem (for example, some form of amplitude modulation) which is too expensive for use in each repeater might profitably be used here where it would appear in only every tenth repeater.

(*iii*) System parameters which have a first order effect on information capacity are the symbol rate, repeater spacing, and cable diameter. On the other hand, the hybrid cable system is relatively insensitive to variations in repeater output power, repeater noise figure and average probability of error.

APPENDIX A

Best Uniform Repeater Spacing

$\hat{\mathcal{E}}$ denotes the minimum value of $\mathcal{E}$ attained by jointly designing the analog and digital repeater characteristics as pointed out in Part I.[1] Some terms in $\hat{\mathcal{E}}$ are extremely small for the coaxial cable systems considered. With such terms neglected, $\hat{\mathcal{E}}$ can be further minimized by using uniform repeater spacing.

Minimum notations are used in the text for clarity, but it is necessary to add a rather large number in the appendices.

Part I showed that under two conditions we have

$$\hat{\mathcal{E}} = \boldsymbol{\beta}'\mathbf{Q}^{-1}\boldsymbol{\beta} \tag{12}$$

where

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_L \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} P_0 + \alpha_{00} & \alpha_{10} & \cdots & \alpha_{L0} \\ \alpha_{01} & P_1 + \alpha_{11} & \cdots & \alpha_{L1} \\ \vdots & \vdots & & \vdots \\ \alpha_{0L} & \alpha_{1L} & \cdots & P_L + \alpha_{LL} \end{bmatrix},$$

$$\beta_l = \int_{-1/2T}^{1/2T} [TM(f)]^{\frac{1}{2}} \frac{\left[N_{l+1}\left(f - \frac{m}{T}\right)\right]^{\frac{1}{2}}}{\left|A_l\left(f - \frac{m}{T}\right)\right|} df, \qquad l = 0, \cdots, L$$

$$\alpha_{hl} = \int_{-1/2T}^{1/2T} \frac{\left[N_{l+1}\left(f - \frac{m}{T}\right)\right]^{\frac{1}{2}}}{\left|A_l\left(f - \frac{m}{T}\right)\right|} \frac{\left[N_{h+1}\left(f - \frac{m}{T}\right)\right]^{\frac{1}{2}}}{\left|A_h\left(f - \frac{m}{T}\right)\right|} df, \quad h, l = 0, \cdots, L.$$

In the above, m is the integer that minimizes the ratios $[N_{l+1}(f - m/T)]^{1/2} / | A_l(f - m/T) |$ (notice that m may vary with f), $1/T$ is the symbol rate, and $M(f)$ is the spectral density of the stationary random message sequence $\{a_k\}$. Furthermore, as illustrated in Fig. 1 of Part I, $N_l(f)$ is the spectral density of the noise at the input of the lth analog repeater $B_l(f)$, $N_{L+1}(f)$ is the spectral density of the noise at the input of the receiving filter $B_{L+1}(f)$, $A_l(f)$ is the transfer function of the transmission medium between $B_l(f)$ and $B_{l+1}(f)$, and P_l is the average output power of $B_l(f)$.

Equation (12) is valid under the two conditions:

(i) For any frequency f and integer K, and for $l = 0, \ldots, L$, we have either

$$\frac{[N_{l+1}(f)]^{\frac{1}{2}}}{| A_l(f) |} > \frac{\left[N_{l+1}\left(f - \frac{K}{T}\right)\right]^{\frac{1}{2}}}{\left| A_l\left(f - \frac{K}{T}\right) \right|}, \quad \text{for all} \quad l \tag{13}$$

or

$$\frac{[N_{l+1}(f)]^{\frac{1}{2}}}{| A_l(f) |} < \frac{\left[N_{l+1}\left(f - \frac{K}{T}\right)\right]^{\frac{1}{2}}}{\left| A_l\left(f - \frac{K}{T}\right) \right|}, \quad \text{for all} \quad l. \tag{14}$$

(ii) Let

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_0^{\frac{1}{2}} \\ \lambda_1^{\frac{1}{2}} \\ \vdots \\ \lambda_L^{\frac{1}{2}} \end{bmatrix} = \mathbf{Q}^{-1}\boldsymbol{\beta} \tag{15}$$

then we must have

$$[TM(f)]^{\frac{1}{2}} - \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} \frac{\left[N_{l+1}\left(f - \frac{m}{T}\right)\right]^{\frac{1}{2}}}{\left| A_l\left(f - \frac{m}{T}\right) \right|} \geqq 0, \quad \text{for} \quad -\frac{1}{2T} \leqq f \leqq \frac{1}{2T} \tag{16}$$

and

$$[\lambda_l]^{\frac{1}{2}} > 0, \qquad l = 0, \cdots, L \tag{17}$$

where m, as defined previously, is the integer that minimizes the ratios $[N_{l+1}(f - m/T)]^{1/2}/|A_l(f - m/T)|$.

Notice that condition i guarantees that m does not vary with l.

Now we consider the coaxial cable systems. We assume in Section II for the coaxial cable systems that

$$E[a_k] = 0, \tag{18}$$

$$E[a_k^2] = \frac{d^2(\nu^2 - 1)}{12}, \tag{19}$$

$$E[a_i a_{i+j}] = 0, \qquad j \neq 0, \tag{20}$$

$$N_l(f) = N_0, \qquad l = 1, \cdots, L + 1, \tag{21}$$

and

$$P_l = P, \qquad l = 0, \cdots, L. \tag{22}$$

By definition[1]

$$M(f) = E[a_k^2] + 2 \sum_{j=1}^{\infty} E[a_i a_{i+j}] \cos 2\pi f j T. \tag{23}$$

Substituting (19) and (20) into (23) gives

$$M(f) = \frac{d^2(\nu^2 - 1)}{12}. \tag{24}$$

In order to consider repeater spacing, let us define

S_l = length of the cable (in miles) between the repeaters $B_l(f)$ and $B_{l+1}(f)$, $l = 0, \ldots, L$. Over most of the useful frequency range one may assume[2]

$$|A_l(f)| = e^{-S_l(|f|/f_0)^{1/2}}, \qquad l = 0, \cdots, L \tag{25}$$

where f_0 is a cable constant.

From (21) and (25),

$$\frac{[N_{l+1}(f)]^{1/2}}{|A_l(f)|} = (N_0)^{1/2} e^{S_l(|f|/f_0)^{1/2}}, \qquad l = 0, \cdots, L. \tag{26}$$

Since the right side of (26) increases monotonically with f, condition i is satisfied (that is, for any f and K either (13) holds for all l, or (14) holds for all l).

Notice from (26) that, in general, the ratios $[N_{l+1}(f)]^{1/2}/|A_l(f)|$ will increase monotonically with f even if the repeater noises are not

white (the variation in the exponent usually outweighs possible variations in the noise spectral densities). Thus, condition i will usually be satisfied even if white noises are not assumed.

The second condition in (16) and (17) serves as a final check. It is not used in any computation, and we only have to show that our results satisfy it. This is done at the end of this appendix.

As already discussed, for each f in $-1/2T \leqq f \leqq 1/2T$, we should choose the integer m in β_l and α_{hl} to minimize the ratios

$$\frac{\left[N_{l+1}\left(f - \frac{m}{T}\right)\right]^{\frac{1}{2}}}{\left|A_l\left(f - \frac{m}{T}\right)\right|}, \qquad l = 0, \cdots, L.$$

It is clear from (26) that these ratios are minimized by choosing

$$m = 0, \qquad \text{for all } f \text{ in } -\frac{1}{2T} \leqq f \leqq \frac{1}{2T}. \tag{27}$$

Substituting (24), (21), (25), and (27) into the definition of β_l gives

$$\beta_l = \psi \frac{1}{\mu_l^2}\left[1 + (\mu_l - 1)e^{\mu_l}\right], \qquad l = 0, \cdots, L \tag{28}$$

where

$$\psi = \left[\frac{N_0\, d^2(\nu^2 - 1)}{3T}\right]^{\frac{1}{2}}$$

$$\mu_l = \frac{S_l}{[2Tf_0]^{\frac{1}{2}}}.$$

Substituting (21), (25), and (27) into the definition of α_{hl} gives

$$\alpha_{hl} = \frac{2N_0}{T(\mu_h + \mu_l)^2}\left[1 + (\mu_h + \mu_l - 1)e^{\mu_h + \mu_l}\right], \qquad h, l = 0, \cdots, L. \tag{29}$$

Substituting (22) into (12) gives

$$\hat{\varepsilon} = \boldsymbol{\beta}'[P\mathbf{I} + \boldsymbol{\alpha}]^{-1}\boldsymbol{\beta} \tag{30}$$

where $\mathbf{I}$ is the identity matrix and

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_{00} & \alpha_{10} & \cdots & \alpha_{L0} \\ \alpha_{01} & \alpha_{11} & \cdots & \alpha_{L1} \\ \vdots & \vdots & & \vdots \\ \alpha_{0L} & \alpha_{1L} & \cdots & \alpha_{LL} \end{bmatrix}.$$

The total distance between the two digital repeaters $B_0(f)$ and $B_{L+1}(f)$ is

$$\eta = S_0 + S_1 + \cdots + S_L . \tag{31}$$

Clearly, we should regard η as fixed when varying the repeater spacings S_0 to S_L to minimize $\hat{\mathcal{E}}$.

It is customary to use uniform repeater spacing, that is,

$$S_l = \frac{\eta}{L+1} = S, \qquad l = 0, \cdots, L \tag{32}$$

where S is the common repeater spacing. It is shown in the following that:

(i) Uniform repeater spacing minimizes the mean square error $\hat{\mathcal{E}}$ if α is negligible in (30).

(ii) α is indeed negligible for the coaxial cable systems considered. (Therefore, uniform repeater spacing is considered in the text.)

Let us prove the first statement. When α is negligible in (30), we have

$$\hat{\mathcal{E}} \cong \mathcal{E}_0 = \boldsymbol{\beta}'[P\mathbf{I}]^{-1}\boldsymbol{\beta} \tag{33}$$

where $\mathcal{E}_0$ is an abbreviation.

Substituting (28) into (33) yields

$$\mathcal{E}_0 = \frac{1}{P}\psi^2 \sum_{l=0}^{L} \frac{1}{\mu_l^4}[1 + (\mu_l - 1)e^{\mu_l}]^2. \tag{34}$$

We now determine the repeater spacings S_0 to S_L which minimize $\mathcal{E}_0$ in (34), subject to the constraint in (31). Since there is a one-to-one correspondence between S_l and μ_l, the problem is equivalent to determining the values of μ_0 to μ_L which minimize $\mathcal{E}_0$ in (34), subject to the constraint

$$\mu_0 + \mu_1 + \cdots + \mu_L = \mu_{\text{total}} \tag{35}$$

where μ_{total} is a fixed constant.

A necessary condition for μ_l to minimize $\mathcal{E}_0$ subject to the constraint in (35) is

$$x_l = \lambda = 0, \qquad l = 0, \cdots, L \tag{36}$$

where λ is a Lagrange multiplier and

$$x_l = \frac{2}{\mu_l^3}[1 + (\mu_l - 1)e^{\mu_l}]e^{\mu_l} - \frac{4}{\mu_l^5}[1 + (\mu_l - 1)e^{\mu_l}]^2. \tag{37}$$

Clearly (36) requires that

$$x_i = x_j, \qquad i, j = 0, \cdots, L; \qquad i \neq j. \tag{38}$$

We may solve (35) and the L equations in (38) for μ_0 to μ_L. Clearly,

$$\mu_l = \frac{\mu_{\text{total}}}{L+1}, \qquad l = 0, \cdots, L \tag{39}$$

is a solution. It is also the only solution because since repeater spacing $S_l > 0$, we have $\mu_l > 0$. It can be shown for all $l = 0, \ldots, L$ that

$$x_l = \tfrac{1}{3} \quad \text{when} \quad \mu_l = 0,$$

and that x_l increases montonically with μ_l when $\mu_l > 0$. From this it is clear that x_l's cannot be all equal as required by (38) if μ_l's are not all equal. Therefore, (39) is the unique solution of the constraint (35) and the necessary conditions (38). It can be easily established that $\mathcal{E}_0$ is a minimum, not a maximum, at (39). Therefore, (39) minimizes $\mathcal{E}_0$, or, uniform repeater spacing minimizes $\mathcal{E}_0$.

Next we must show that α is so small that for all practical purposes minimizing $\mathcal{E}_0$ minimizes $\hat{\mathcal{E}}$. Notice that it is not necessary to show this for all possible repeater spacings. Clearly we do not have to show this for classes of nonuniform repeater spacings which we know will produce $\hat{\mathcal{E}}$ larger than the $\hat{\mathcal{E}}$ of uniform repeater spacing. One such class is that which calls for

$$S_l > YS, \qquad \text{for at least one } l,$$

or equivalently

$$\mu_l > Y\mu, \qquad \text{for at least one } l,$$

where

$$\mu = \frac{S}{(2Tf_0)^{\frac{1}{2}}}, \tag{40}$$

and Y is given by the equality

$$\frac{\dfrac{Td^2(\nu^2-1)}{3(Y\mu)^4T^2}\left[(Y\mu-1)e^{Y\mu}+1\right]^2}{\dfrac{P}{N_0}+\dfrac{1}{2(Y\mu)^2T}\left[(2Y\mu-1)e^{2Y\mu}+1\right]} = \frac{\dfrac{Td^2(\nu^2-1)(L+1)}{3\mu^4T^2}\left[(\mu-1)e^{\mu}+1\right]^2}{\dfrac{P}{N_0}+\dfrac{L+1}{2\mu^2T}\left[(2\mu-1)e^{2\mu}+1\right]}. \tag{41}$$

The right side of (41) is the $\hat{\mathcal{E}}$ of uniform repeater spacing (see Appendix C). The left side can be easily shown to be a lower bound of $\hat{\mathcal{E}}$ for the class of nonuniform repeater spacings which calls for $S_l > YS$ for at least one l. Hence, this class produces an $\hat{\mathcal{E}}$ larger than tbat of uniform repeater spacing, and should be ruled out. Consequently, it is only necessary to show that $\boldsymbol{\alpha}$ is negligible when

$$S_l \leqq YS, \qquad \text{for all } l = 0, \cdots, L. \tag{42}$$

This can be easily shown for the coaxial cable systems considered. For example, consider tbe typical system parameter values (Section III): $f_0 = 5 \times 10^6$ hertz, $P = 0.1$ watt, $N_0 = 10^{-19}$ watts per hertz, $S = 1.25$ miles, $L = 9$, and $1/T = 2.8 \times 10^8$. Substituting these values into (41) gives

$$y = 1.2.$$

Notice that it is not necessary to specify d and ν because they cancel out in (41). Furthermore, it can be shown that the left side of (41) increases monotonically with Y under the conditions in (16) and (17). Hence, the solution of Y is unique. Substituting the above values into (42) and (29), one can show that the largest element in $\boldsymbol{\alpha}$ cannot exceed 0.00003, which is extremely small compared with 0.1 for the diagonal elements of PI. Thus, $\boldsymbol{\alpha}$ is negligible in (30), and, for all practical purposes, minimizing $\mathcal{E}_0$ minimizes $\hat{\mathcal{E}}$.

Finally, before adopting uniform repeater spacing, we must show that it satisfies (16) and (17) in condition *ii*. As shown in Appendix C, for uniform repeater spacing, (17) is automatically satisfied and (16) is equivalent to (6). As discussed in Section II and demonstrated in Section III, (6) is easily satisfied. Therefore, uniform repeater spacing easily satisfies (16) and (17) in condition *ii*.

APPENDIX B

Optimum Repeater Characteristics

Part I showed that,[1] under the same conditions—(13) to (17) in Appendix A—the analog and digital repeater characteristics which minimize the mean square error $\mathcal{E}$ are:

$$\begin{aligned} &| B_0(f) |^2 \\ &= \frac{T[N_1(f)]^{\frac{1}{2}}}{M(f) \mid A_0(f) \mid \lambda_0^{\frac{1}{2}}} \left\{ [TM(f)]^{\frac{1}{2}} - \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} \frac{[N_{l+1}(f)]^{\frac{1}{2}}}{\mid A_l(f) \mid} \right\} \quad \text{for} \quad f \in \mathfrak{F}, \\ &= 0, \quad \text{for} \quad f \notin \mathfrak{F} \end{aligned} \tag{43}$$

$$| B_j(f) |^2 = \left[\frac{\lambda_{j-1} N_{j+1}(f)}{\lambda_j N_j(f)}\right]^{\frac{1}{2}} \frac{1}{| A_{j-1}(f) A_j(f) |}, \quad \text{for} \quad f \varepsilon \mathfrak{F},$$

$$= \text{arbitrary}, \quad \text{for} \quad f \not\varepsilon \mathfrak{F}, \qquad j = 1, 2, \cdots, L \tag{44}$$

$$| B_{L+1}(f) |^2 = \left[\frac{\lambda_0 \lambda_L}{N_1(f) N_{L+1}(f)}\right]^{\frac{1}{2}} \frac{| A_0(f) |}{| A_L(f) |} \frac{M(f)}{T} | B_0(f) |^2, \quad \text{for all} \quad f \tag{45}$$

and the repeater phases need only satisfy the condition

$$e^{-i\theta(f)} = 1, \quad \text{for} \quad f \varepsilon \mathfrak{F},$$

$$= \text{arbitrary}, \quad \text{for} \quad f \not\varepsilon \mathfrak{F} \tag{46}$$

where $\theta(f)$ is the over-all phase of the system that is defined by

$$\prod_{i=0}^{L+1} A_i(f) B_i(f) = \left| \prod_{i=0}^{L+1} A_i(f) B_i(f) \right| e^{-i\theta(f)}.$$

A time delay may always be added to $\theta(f)$. Furthermore, $\theta(f)$ may be distributed arbitrarily among the repeaters. The notations above are all defined in Appendix A after (12) except $\mathfrak{F}$ which is the frequency set

$$\mathfrak{F} = \left\{ f : f = g - \frac{m}{T}, -\frac{1}{2T} \leqq g \leqq \frac{1}{2T} \right\} \tag{47}$$

where m is also defined after (12).

Now apply these general equations to the coaxial cable systems considered. Substituting m in (27) into (47) gives

$$\mathfrak{F} = \left\{ f : -\frac{1}{2T} \leqq f \leqq \frac{1}{2T} \right\}. \tag{48}$$

The best uniform repeater spacing has been discussed in Appendix A. In this and the following appendices, we adopt uniform repeater spacing. Therefore,

$$S_l = S, \qquad l = 0, \cdots, L \tag{49}$$

and

$$| A_l(f) | = e^{-S(|f|/f_0)^{\frac{1}{2}}}, \qquad l = 0, \cdots, L. \tag{50}$$

Substituting (49) into (28) and (29) gives

$$\beta_l = \psi \frac{1}{\mu^2} [1 + (\mu - 1)e^{\mu}], \qquad l = 0, \cdots, L \tag{51}$$

$$\alpha_{hl} = \frac{N_0}{2\mu^2 T} [1 + (2\mu - 1)e^{2\mu}], \qquad h, l = 0, \cdots, L \tag{52}$$

where

$$\mu = \frac{S}{[2Tf_0]^{\frac{1}{2}}}.$$

Notice that now β_l and α_{hl} do not vary with h and l. Substituting (51), (52), and (22) into (15), one obtains

$$[\lambda_l]^{\frac{1}{2}} = \frac{\psi \frac{1}{\mu^2} [1 + (\mu - 1)e^{\mu}]}{P + (L + 1) \frac{N_0}{2\mu^2 T} [1 + (2\mu - 1)e^{2\mu}]} \qquad l = 0, \cdots, L. \quad (53)$$

Now λ_l does not vary with l.

Substituting (21), (24), (48), (50), and (53) into (43) to (45) gives the repeater amplitude characteristics which minimize the mean square error as:

$$| B_0(f) | = \left[\frac{12TN_0^{\frac{1}{2}}e^{S(f/f_0)^{\frac{1}{2}}}}{d^2(\nu^2 - 1)}\right]^{\frac{1}{2}} \left\{\left[\frac{Td^2(\nu^2 - 1)}{12\lambda_0}\right]^{\frac{1}{2}} - (L + 1)N_0^{\frac{1}{2}}e^{S(f/f_0)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}, \quad \text{for} \quad 0 \leqq f \leqq \frac{1}{2T},$$

$$= 0, \quad \text{for} \quad f > \frac{1}{2T} \qquad (54)$$

$$| B_{L+1}(f) | = \left[\frac{\lambda_0\, d^2(\nu^2 - 1)}{12TN_0}\right]^{\frac{1}{2}} | B_0(f) |, \quad \text{for all} \quad f \qquad (55)$$

and

$$| B_j(f) | = e^{S(f/f_0)^{\frac{1}{2}}}, \qquad 0 \leqq f \leqq \frac{1}{2T},$$

$$= \text{arbitrary}, \qquad f > \frac{1}{2T} \qquad j = 1, 2, \cdots, L. \qquad (56)$$

Several observations are made from (54) to (56). First, $| B_0(f) |$ and $| B_{L+1}(f) |$ differ only by a multiplicative constant. Therefore, identical filters may be used for the transmitting and receiving filters in the digital repeaters. [As discussed after (46), an all-pass network may be used at any point of the system to adjust overall phase of the system.]

Second, the $|B_j(f)|$, $j = 1, \ldots, L$, do not vary with j; hence, identical analog repeaters may be used. Furthermore, $|B_j(f)|$ in (56) is just the reciprocal of $|A_l(f)|$ in (50); therefore the analog repeater

gain-frequency characteristic is shaped to match the loss-frequency characteristic of the coaxial cable.

Third, it is much more difficult to realize the transmitting and receiving filters in the digital repeaters than it is the analog repeater filters. This is because $|B_0(f)|$ and $|B_{L+1}(f)|$ must be zero for $f > 1/2T$ (usually requiring a vertical cutoff, or discontinuity, at $f = 1/2T$), while the analog repeater filters $B_1(f)$ to $B_L(f)$ may have arbitrary amplitudes for $f > 1/2T$ (no discontinuity is required at $f = 1/2T$ and the filters may cut off in any convenient manner).

APPENDIX C

Information Rate

Substituting β_l in (51), α_{hl} in (52), and (22) into (12) yields

$$\hat{\varepsilon} = \frac{T \dfrac{d^2(\nu^2 - 1)}{3} \dfrac{1}{\mu^4 T^2} [(\mu - 1)e^{\mu} + 1]^2}{\dfrac{P}{N} + \dfrac{1}{2\mu^2 T} [(2\mu - 1)e^{2\mu} + 1]} \tag{57}$$

where

$$N = (L + 1)N_0 .$$

Solving (57) for ν^2, yields

$$\nu^2 = 1 + \frac{\dfrac{S^2 P}{f_0 N} + (2\mu - 1)e^{2\mu} + 1}{\dfrac{2}{3\mu^2} [(\mu - 1)e^{\mu} + 1]} \cdot \frac{\hat{\varepsilon}}{d^2}. \tag{58}$$

The bit rate R is therefore given by

$$\begin{aligned} R &= \frac{1}{T} \log_2 \nu \\ &= \frac{1}{2T} \log_2 \left\{ 1 + \frac{\dfrac{S^2 P}{f_0 N} + (2\mu - 1)e^{2\mu} + 1}{\dfrac{2}{3\mu^2} [(\mu - 1)e^{\mu} + 1]^2} \cdot \frac{\hat{\varepsilon}}{d^2} \right\}. \end{aligned}$$

which is (7) in Section II.

Equation (12) and hence (7) are valid under the conditions stated in (13), (14), (16), and (17). Appendix A showed that the condition in (13) and (14) is satisfied. Furthermore, it can be easily shown

from (53) that (17) is satisfied. Hence, the only condition remaining is (16). From (21), (24), (27), and (50), it is clear that now (16) is satisfied if and only if it is satisfied at $f = 1/2T$, or (rearranging the terms) if and only if

$$\frac{P}{N_0} \geqq \frac{L+1}{2T\mu^2}\left[2\mu e^{2\mu} - 3e^{2\mu} + 4e^{\mu} - 1\right]$$

which is condition (6).

APPENDIX D

Symbol and Bit Rates

This appendix proves that under the normal operating condition in (6), the number of bits per symbol, $\log_2 \nu$, decreases monotonically when the baud rate increases.

Clearly, $\log_2 \nu$ decreases if ν^2 in (58) decreases. For a given system, the quantities S, f_0, P, L, and N_0 do not vary with the baud rate $1/T$. The quantity $\hat{\varepsilon}/d^2$ is fixed to obtain an approximately constant error rate. The quantity μ, however, is a function of $1/T$. Therefore, we have

$$\frac{\partial(\nu^2)}{\partial T} = \frac{\partial(\nu^2)}{\partial\mu}\frac{\partial\mu}{\partial T}$$

$$= \frac{\hat{\varepsilon}}{d^2}\frac{\partial}{\partial\mu}\left\{\frac{\dfrac{S^2P}{f_0N} + (2\mu - 1)e^{2\mu} + 1}{\dfrac{2}{3\mu^2}\left[(\mu - 1)e^{\mu} + 1\right]^2}\right\}\frac{\partial}{\partial T}\frac{S}{(2Tf_0)^{\frac{1}{2}}}$$

$$= -\frac{\hat{\varepsilon}}{d^2}\frac{\mu^2}{2T}(\mu e^{\mu} - e^{\mu} + 1)^{-3}\cdot\alpha_1 \tag{59}$$

where

$$\alpha_1 = \left[-3\mu^2e^{\mu} + 3\mu e^{\mu} - 3e^{\mu} + 3\right]\frac{S^2P}{f_0N} + \alpha_2 \tag{60}$$

$$\alpha_2 = 3\mu^2e^{3\mu} + 6\mu^2e^{2\mu} - 3\mu^2e^{\mu}$$
$$- 9\mu e^{3\mu} + 6\mu e^{2\mu} + 3\mu e^{\mu} + 3e^{3\mu} - 3e^{2\mu} - 3e^{\mu} + 3. \tag{61}$$

Clearly, $\mu = S/(2Tf_0)^{1/2} > 0$. It can be shown that

$$\mu e^{\mu} - e^{\mu} + 1 > 0 \quad \text{for} \quad \mu > 0.$$

Therefore, from (59), if $\alpha_1 < 0$ for $\mu > 0$, then $\partial\nu^2/\partial T > 0$ and $\log_2\nu$ decreases when baud rate $1/T$ increases.

We now prove that, under the normal operating condition in (6), $\alpha_1 < 0$ for $\mu > 0$. It can be shown that

$$-3\mu^2 e^\mu + 3\mu e^\mu - 3e^\mu + 3 < 0 \tag{62}$$

for $\mu > 0$. The condition in (6) can be written as

$$\frac{S^2P}{f_0N} \geqq 2\mu e^{2\mu} - 3e^{2\mu} + 4e^\mu - 1. \tag{63}$$

From (62) and (63)

$$\frac{S^2P}{f_0N} [-3\mu^2 e^\mu + 3\mu e^\mu - 3e^\mu + 3] + \alpha_3 \leqq 0 \tag{64}$$

where

$$\alpha_3 = [2\mu e^{2\mu} - 3e^{2\mu} + 4e^\mu - 1][3\mu^2 e^\mu - 3\mu e^\mu + 3e^\mu - 3].$$

It can be shown that

$$\alpha_2 < \alpha_3 \quad \text{for } \mu > 0 \tag{65}$$

Therefore, from (60), (65), and (64)

$$\alpha_1 = [-3\mu^2 e^\mu + 3\mu e^\mu - 3e^\mu + 3] \frac{S^2P}{f_0N} + \alpha_2$$

$$< [-3\mu^2 e^\mu + 3\mu e^\mu - 3e^\mu + 3] \frac{S^2P}{f_0N} + \alpha_3 \leqq 0 \quad \text{for} \quad \mu > 0. \tag{66}$$

Inequality (66) shows that $\alpha_1 < 0$ for $\mu > 0$. From previous discussion, this implies that the number of bits per symbol, $\log_2 \nu$, decreases monotonically when symbol rate $1/T$ increases. The proof is complete.

APPENDIX E

Error Rates

The system from one digital repeater to the next (including L analog repeaters—see Fig. 1) is referred to as a "digital repeater section" in the following.

In case *i* of Section IV, $L = 0$ and each digital repeater section covers a distance of 1.25 miles. There are 100 digital repeater sections in 125 miles. If $d^2/\hat{\varepsilon}$ is set to 144, error rate is approximately 10^{-9} for each digital repeater section, or approximately 10^{-7} over a distance of 125 miles.

In case *ii*, $L = 9$ and each digital repeater section covers 12.5 miles.

If $d^2/\hat{\mathcal{E}}$ is set to 126, error rate is about 10^{-8} for each digital repeater section, or approximately 10^{-7} over a distance of 125 miles.

In case *iii*, $L = 99$ and each digital repeater section covers 125 miles. If $d^2/\hat{\mathcal{E}}$ is set to 108, error rate is approximately 10^{-7} for each digital repeater section, that is, a distance of 125 miles.

The bit rates in Table I are not sensitive to variations in $d^2/\hat{\mathcal{E}}$. For instance, if one sets $d^2/\hat{\mathcal{E}}$ to 126 for all three cases (comparing the three cases with the same mean square error at decision circuit inputs), the bit rate of case *i* increases only about 1.5 percent from that in Table I, the bit rate of case *ii* is unchanged, and the bit rate of case *iii* decreases only about 2 percent.

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